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ON A CERTAIN TYPE OF LOWER HESSENBERG-TOEPLITZ MATRICES AND RAMANUJAN'S TAU FUNCTION

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Abstract. We exhibit the deduction of a formula for the characteristic polynomials of certain lower Hessenberg-Toeplitz matrices, and thus for the case of Ramanujan's tau function we prove the corresponding expression recently conjectured by Prof. Barry Brent via computer data. Besides, we indicate that the Cayley-Hamilton theorem applied to these matrices gives the same information than a recurrence relation involving their determinants. We show that the complete Bell polynomials allow obtain the generating function for these characteristic polynomials.

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1. Introduction

Here we study the following type of lower Hessenberg-Toeplitz matrices [1]:

(1)
$$\mathbf{A}_r = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r-1} & a_{r-2} & \cdots & a_1 & r-1 \\ a_r & a_{r-1} & \cdots & a_2 & a_1 \end{pmatrix}, \quad A_r := \det \mathbf{A}_r,$$

with special interest in their characteristic polynomials:

$$P_r(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}_r), \quad P_0(\lambda) = 1. \tag{2}$$

In Sec. 2 we show a simple process to obtain an explicit formula for (2) in terms of the determinants A_r, which may be applied to Ramanujan's tau function [2–5] to prove a recent result that Prof. Barry Brent [6] conjectured via computer data. Besides, we indicate that the Cayley-Hamilton theorem for (1) has the same information than the recurrence relation:

$$A_r = (r-1)! \sum_{j=1}^{r} \frac{(-1)^{j-1}}{(r-j)!} a_j A_{r-j}, \quad r > 1, \quad A_0 = 1, \tag{3}$$

which gives the known relation satisfied by the tau function [2–5]:

$$n\tau(n+1) = -24 \sum_{j=1}^{n} \sigma(j)\tau(n+1-j), \quad n \geq 1, \tag{4}$$

involving the sum of divisors function [7–11]. In Sec. 3 we use the generating function of complete Bell polynomials [12–18] to obtain the corresponding generating function for the characteristic polynomials (2).

2. An expression for $P_r(\lambda)$

In (2) we can give values to r and realize a direct computation of characteristic polynomials, thus:

$$\begin{aligned}
&P_1 = -a_1 + \lambda, \quad P_2 = a_1^2 - a_2 - 2a_1\lambda + \lambda^2,\\
&P_3 = -a_1^3 + 3a_1a_2 - 2a_3 + 3\left(a_1^2 - a_2\right)\lambda\\
&\quad - 3a_1\lambda^2 + \lambda^3,\\
&P_4 = a_1^4 - 6a_1^2a_2 + 8a_1a_3 + 3a_2^2 - 6a_4\\
&\quad + 4\left(-a_1^3 + 3a_1a_2 - 2a_3\right)\lambda\\
&\quad + 6\left(a_1^2 - a_2\right)\lambda^2 - 4a_1\lambda^3 + \lambda^4,\\
&P_5 = -a_1^5 + 10a_1^3a_2 - 20a_1^2a_3 - 15a_1a_2^2\\
&\quad + 30a_1a_4 + 20a_2a_3 - 24a_5\\
&\quad + 5(a_1^4 - 6a_1^2a_2 + 8a_1a_3 + 3a_2^2 - 6a_4)\lambda\\
&\quad + 10(-a_1^3 + 3a_1a_2 - 2a_3)\lambda^2\\
&\quad + 10(a_1^2 - a_2)\lambda^3 - 5a_1\lambda^4 + \lambda^5, \ldots,
\end{aligned} \tag{5}$$

and it is clear that $A_r = (-1)^r P_r(0)$, then (5) can be written in the form:

$$\begin{aligned}
&P_0 = A_0, \quad P_1 = -A_1 + \lambda,\\
&P_2 = A_2 - 2A_1\lambda + \lambda^2,\\
&P_3 = -A_3 + 3A_2\lambda - 3A_1\lambda^2 + \lambda^3,\\
&P_4 = A_4 - 4A_3\lambda + 6A_2\lambda^2 - 4A_1\lambda^3 + \lambda^4,\\
&P_5 = -A_5 + 5A_4\lambda - 10A_3\lambda^2 + 10A_2\lambda^3\\
&\quad - 5A_1\lambda^4 + \lambda^5, \ldots,
\end{aligned} \tag{6}$$

that is:

$$P_r(\lambda) = (\lambda - \phi)^r, \quad \phi^k := A_k, \tag{7}$$

therefore:

$$P_r(\lambda) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} A_{r-k} \lambda^k, \tag{8}$$

which is correct [1].

For the case $a_j = 24\sigma(j)$, we have [6] the value $A_j = (-1)^j j! \tau(j+1)$, then (8) implies the expression:

$$P_m(\lambda) = \sum_{k=0}^{m} \frac{m!}{k!} \tau(m-k+1) \lambda^k, \tag{9}$$

conjectured by Brent [6] via computer data.

The Cayley-Hamilton theorem indicates that $\mathbf{A}_r$ must verify its characteristic polynomial (8), then:

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} A_{r-k} \mathbf{A}_r^k = \mathbf{0}, \tag{10}$$

where we can employ (1) to obtain the same relations deduced from the recurrence relation (3); so, the property (4) is immediate from (3) if $a_m = 24\sigma(m)$ and $A_m = (-1)^m m! \tau(m+1)$.

If we define the quantities $Q_j = \frac{(-1)^j}{j!} A_j$, then (3) acquires the form:

$$r Q_r = -\sum_{j=1}^{r} a_j Q_{r-j}, \quad r \geq 1, \tag{11}$$

which is attractive because we know its solution [18–20]:

$$Q_r = \frac{1}{r!} B_r(-0!a_1, -1!a_2, \ldots, -(r-1)!a_r), \tag{12}$$

in terms of complete Bell polynomials [12–18]; many important arithmetic functions satisfy recurrence relations with the structure (11) [18, 21–27]. Hence:

$$A_r = (-1)^r B_r(-0!a_1, \ldots, -(r-1)!a_r), \tag{13}$$

and then the Ramanujan's tau function admits the following representation [5]:

(14)
$$\tau(r+1) = \frac{1}{r!} B_r\left(-0!24\sigma(1), -1!24\sigma(2), \ldots, -(r-1)!24\sigma(r)\right).$$

3. Generating functions for $P_r(\lambda)$

We know the generating function for the complete Bell polynomials:

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, \ldots, x_n) \frac{t^n}{n!} = \exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right), \tag{15}$$

which is useful to obtain the corresponding generating function for the characteristic polynomials (2), in fact, from (8) and (13):

$$P_r(\lambda) = \sum_{k=0}^{r} \binom{r}{k} B_k\left(-0!a_1, \ldots, -(k-1)!a_k\right) \lambda^{r-k}, \tag{16}$$

therefore:

$$\begin{aligned}
\sum_{r=0}^{\infty} P_r(\lambda)\frac{t^r}{r!} &= \sum_{k=0}^{\infty} \frac{1}{k!} B_k\left(\right) \sum_{r=k}^{\infty} \frac{\lambda^{r-k} t^r}{(r-k)!} \\
&= \sum_{k=0}^{\infty} B_k\left(\right) \frac{t^k}{k!} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \\
&= e^{\lambda t} \exp\left(\sum_{m=1}^{\infty} \frac{-(m-1)!a_m}{m!} t^m\right) \\
&= \exp\left(\lambda t - \sum_{m=1}^{\infty} \frac{a_m}{m} t^m\right),
\end{aligned} \tag{17}$$

where we use $a_m = 24\sigma(m)$ if we work with the Ramanujan's tau function. Here we see the following option: To employ a different procedure to show (17), and after with it perform the above process in reverse to deduce (8) and (13); in another paper we will develop in detail this alternative method to show (8).

Remark 1. *The characteristic polynomials (5) verify the recurrence relation:*

$$P_{r+1}(\lambda) = \lambda P_r(\lambda) - \sum_{k=0}^{r} \frac{r!}{k!} a_{r-k+1} P_k(\lambda), \quad r \geq 0. \tag{18}$$

Remark 2. *The property (13) accepts the inversion [28]:*

$$a_r = \frac{1}{(r-1)!} \sum_{k=1}^{r} (-1)^k (k-1)! B_{r,k} \left(-A_1, A_2, -A_3, \ldots, (-1)^{r-k+1} A_{r-k+1} \right) \tag{19}$$

in terms of partial Bell polynomials [28, 29].

Remark 3. *The symbolic formula (7) was suggested by a similar expression for Laguerre polynomials [30]:*

$$L_r(x) = (1-\phi)^r, \quad \phi^k := \frac{x^k}{k!} \tag{20}$$

Remark 4. *The characteristic polynomial (8) has coefficients involving the determinants A_r. On the other hand, the Leverrier-Takeno method [31] employs the traces of $\mathbf{A}_r^k$ to determine these coefficients, then it is natural to investigate the connection between A_j and these traces, in fact:*

$$A_r = \frac{(-1)^r}{r!} B_r \left(-0! s_1, -1! s_2, \ldots, -(r-1)! s_r \right), \quad s_k = tr \left(\mathbf{A}_r^k \right) \tag{21}$$

which can be compared with (13).

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